Math 213-031 **Final Exam** **8/05/19**

Directions: Read each problem carefully. Number each problem. Show all work leading to an answer. Answer all parts of each question. Clearly label your answers. No Calculators!

1. (8 points) Evaluate the line integral $\int_C \left(xy + y^2 + z\right)ds$ over the curve C: the line segment from $(0, 0, 2)$ to $(2, 3, 1)$.

$\vec{r}(t) = < 2t, 3t, 2 - t >,$ $\quad ds = \left|\frac{d\vec{r}}{dt}\right|dt = \left|\vec{v}\right|dt = \sqrt{14}dt$

$$\int_C \left(xy + y^2 + z\right)ds = \int_0^1 \left((2t)(3t) + (3t)^2 + (2 - t)\right)\sqrt{14}dt = \sqrt{14}\int_0^1 \left(15t^2 - t + 2\right)dt$$

$$= \frac{13\sqrt{14}}{2}$$

2. (20 points) Let $\vec{F} = xy\vec{i} + x^2y^3\vec{j}$ and let C be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

a. Directly compute the counterclockwise circulation of $\vec{F}$ around C.

$$Circ = \int_C \vec{F}\cdot\vec{T}ds = \int_{C_1} \vec{F}\cdot\vec{T}ds + \int_{C_2} \vec{F}\cdot\vec{T}ds + \int_{C_3} \vec{F}\cdot\vec{T}ds$$

$$= 0 + \int_0^1 < (1)(2t), (1)^2(2t)^3 > \cdot < 0, 2 > dt + \int_0^1 < (1 - t)(2 - t), (1 - t)^2(2 - 2t)^3 > \cdot <$$

$$= \int_0^1 16t^3dt + \int_0^1 \left(-2(1 - t)^2 - 16(1 - t)^5\right)dt = 4 - \frac{10}{3} = \frac{2}{3}$$

b. Verify the answer from part a. by using Green's Theorem to compute the circulation of $\vec{F}$ around C.

$$Circ = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \iint_R \left(\frac{\partial}{\partial x}\left(x^2y^3\right) - \frac{\partial}{\partial y}(xy)\right)dA$$

$$= \int_0^1\int_0^{2x} \left(2xy^3 - x\right)dydx = \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_0^{2x} dx = \int_0^1 [8x^5 - 2x^3]dx = \ldots = \frac{2}{3}$$

3. (20 points) Let $\vec{F} = -x^2y\vec{i} + xy^2\vec{j}$ and let C be the circle $x^2 + y^2 = 4$.

a. Directly compute the flux of $\vec{F}$ across C.

$\vec{r}(t) = < 2\cos\cos t, 2\sin\sin t > = < x, y >,$

$\frac{d\vec{r}}{dt} = < -2\sin\sin t, 2\cos\cos t > = < dx, dy >$

$$Flux = \oint_C \vec{F}\cdot\vec{N}ds = \oint_C Mdy - Ndx$$

$$= \int_0^{2\pi} \left(-(2\cos\cos t)^2(2\sin\sin t)\right)(2\cos\cos t\, dt) - \left((2\cos\cos t)(2\sin\sin t)^2\right)(-2\sin\sin t$$

$$= 16\int_0^{2\pi}(-\ t\ \sin sin\, t\ +\ t\ \cos cos\, t\ )dt\ =\ 16\left[\frac{t}{4}+\frac{t}{4}\right]_0^{2\pi}\ =\ 0$$

b. Verify the answer from part a. by using Green's Theorem to verify the flux of $\vec{F}$ across C.

$$Flux\ =\ \iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dA\ =\ \iint_R\left(\frac{\partial}{\partial x}\left(-\ x^2y\right)+\frac{\partial}{\partial y}\left(xy^2\right)\right)dA$$

$$=\ \iint_R(-\ 2xy\ +\ 2xy)dA\ =\ \iint_R 0dA\ =\ 0$$

4. (16 points) Let $\vec{F}\ =\ 2xy\vec{i}+\left(x^2\ +\ 2yz\right)\vec{j}+y^2\vec{k}.$

a. Determine if $\vec{F}$ is conservative on R^3.

$$curl\,\vec{F}\ =\ \nabla\ \times\ \vec{F}\ =<\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z}-\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}>$$

$$=<\frac{\partial}{\partial y}\left(y^2\right)-\frac{\partial}{\partial z}\left(x^2\ +\ 2yz\right),\ \frac{\partial}{\partial z}(2xy)-\frac{\partial}{\partial x}\left(y^2\right),\frac{\partial}{\partial x}\left(x^2\ +\ 2yz\right)-\frac{\partial}{\partial y}(2xy)>$$

$=<\ 2y\ -\ 2y, 0\ -\ 0, 2x\ -\ 2x\ >=\ \vec{0}$, so $\vec{F}$ is conservative

b. Find the potential function f of $\vec{F}$.

$$f(x,y,z)=\int\frac{\partial f}{\partial x}dx\ =\ \int Mdx\ =\ \int(2xy)dx\ =\ x^2y\ +\ g(y,z)$$

$$\frac{\partial f}{\partial y}\ =\ \frac{\partial}{\partial y}\left(x^2y\ +\ g(y,z)\right)=x^2\ +\ \frac{\partial g}{\partial y}\ =\ N\ =\ x^2\ +\ 2yz$$

$$g(y,z)=\int\frac{\partial g}{\partial y}dy\ =\ \int 2yzdy\ =\ y^2z\ +\ h(z)$$

$$\frac{\partial f}{\partial z}\ =\frac{\partial}{\partial z}\left(x^2y\ +y^2z\ +\ h(z)\right)=y^2\ +\ \frac{dh}{dz}\ =\ P\ =y^2\ h(z)=\int\frac{dh}{dz}dz\ =\int 0dz\ =\ C$$

$$f(x,y,z)=x^2y\ +y^2z\ +\ C$$

c. Evaluate $\int_C\vec{F}\cdot d\vec{r}$ where C parametrized by

$\vec{r}(t)=(4\sin sin\, t\ )\vec{i}+(\cos cos\, t\ )\vec{j}+\ 3t\vec{k}$, for $0\leq t\leq\frac{\pi}{3}$.

$$\int_C\vec{F}\cdot d\vec{r}\ =\ f\left(\vec{r}\left(\frac{\pi}{3}\right)\right)-\ f\left(\vec{r}(0)\right)=\ f\left(2\sqrt{3},\frac{1}{2},\pi\right)-\ f(0,1,0)=\ 6\ +\frac{\pi^2}{4}$$

5. (16 points) Consider the surface S that is the cap of the paraboloid $z\ =\ 6\ -\ x^2\ -\ y^2$ above $z\ =\ 2$.

a. Find a parametrization of the surface. Include the parameter domain.

Cylindrical: $\vec{r}(u,v)\ =<\ u\cos cos\, v\ ,u\sin sin\, v\ ,6\ -\ u^2\ >,\quad 0\leq u\leq 2,\ 0\leq v\leq 2\pi$

b. Find the surface area of S.

$$SA\ =\ \iint_S d\sigma\ =\ \iint_R\left|\vec{r}_u\ \times\ \vec{r}_v\right|dA$$

$$=\ \iint_R|<\ \cos cos\, v\ ,\sin sin\, v\ ,-\ 2u\ >\times<-\ u\sin sin\, v\ ,u\cos cos\, v\ ,0\ >|dA$$

$= \iint_R \left| < 2u^2 \cos \cos v, 2u^2 \sin \sin v, u > \right| dA = \iint_R u\sqrt{4u^2 + 1}dA$

$= \int_0^{2\pi} \int_0^2 u\sqrt{4u^2 + 1}dudv = \int_0^{2\pi} \left[\frac{1}{8}\frac{2}{3}(4u^2 + 1)^{\frac{3}{2}} \right]_0^2 dv = \frac{1}{12}\int_0^{2\pi} \left[17^{\frac{3}{2}} - 1 \right] dv$

$= \frac{\pi}{6}\left[17^{\frac{3}{2}} - 1 \right]$

c. Evaluate $\iint_S \sqrt{1 + 4x^2 + 4y^2}d\sigma$.

$\iint_S \sqrt{1 + 4x^2 + 4y^2}d\sigma = \iint_R \sqrt{1 + 4x^2 + 4y^2}\left| \vec{r}_u \times \vec{r}_v \right| dA$

$= \iint_R \sqrt{1 + 4u^2}u\sqrt{4u^2 + 1}dA = \int_0^{2\pi} \int_0^2 (4u^3 + u)dudv = \ldots = 36\pi$

6. (10 points) Let $\vec{F} = (y - z)\vec{i} + (z - x)\vec{j} + (x + z)\vec{k}$ and let C be the boundary of the triangle cut from the plane $2x + 3y + z = 6$ by the coordinate planes. Use an appropriate method to compute the circulation of $\vec{F}$ around C.

$\nabla \times \vec{F} = < \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} >$

$= < \frac{\partial}{\partial y}(x + z) - \frac{\partial}{\partial z}(z - x), \frac{\partial}{\partial z}(y - z) - \frac{\partial}{\partial x}(x + z), \frac{\partial}{\partial x}(z - x) - \frac{\partial}{\partial y}(y - z) >$

$= < 0 - 1, -1 - 1, -1 - 1 > = < -1, -2, -2 >$

$Circulation = \oint_C \vec{F} \cdot \vec{T}ds = \iint_S \nabla \times \vec{F} \cdot \vec{n}d\sigma = \iint_S \nabla \times \vec{F} \cdot \vec{n}d\sigma$

$= -\iint_R < -1, -2, -2 > \cdot < 2, 3, 6 > dA = \iint_R 10dA = 10\int_0^3 \int_0^{2 - \frac{2}{3}x} dydx = \ldots = 30\sqrt{14}$

7. (10 points) Let $\vec{F} = \left(x^2 + yz\right)\vec{i} + \left(x^3 - 2xy - 2y^2\right)\vec{j} + (4yz + 1)\vec{k}$ and let S be the boundary of the region between the hyperbolic paraboloid $z = y^2 - x^2$, the cylinder $x^2 + y^2 = 9$, and the plane $z = 10$. Use an appropriate method to compute the outward flux of $\vec{F}$ across S.

$Flux = \iint_S \vec{F} \cdot \vec{n}d\sigma = \iiint_D \nabla \cdot \vec{F}dV$

$= \iiint_D \left(\frac{\partial}{\partial x}\left(x^2 + yz\right) + \frac{\partial}{\partial y}\left(x^3 - 2xy - 2y^2\right) + \frac{\partial}{\partial z}(4yz - 1) \right) dV$

$= \iiint_D ((2x) + (-2x - 4y) + (4y))dV = \iiint_D (0)dV = 0$